M332/SP

SPECIMEN

THE OPEN UNIVERSITY

Third Level Course Examination

COMPLEX ANALYSIS

Specimen Paper

Time allowed: 3 hours

There are **THREE** parts to this paper. In Part I you should attempt as many questions as you can. You should attempt not more than **TWO** questions in Part II and not more than **ONE** question in Part III.

You are advised to spend approximately $1\frac{1}{2}$ *hours* on Part I (50% of total marks), 1 *hour* on Part II (34% of total marks) and $\frac{1}{2}$ *hour* on Part III (16% of total marks).

Write your answers to Part I in one of the answer books provided. Write your answers to Parts II and III in a *separate* answer book. Additional answer books may be used if necessary, still keeping Part I separate from Parts II and III.

At the end of the examination

Check that you have written your name, personal identifier and examination number on **each** answer book used—**Failure to do so will mean that your papers cannot be identified.**

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PART I

Instructions

(i) You should attempt as many questions as you can in this part of the examination.

(ii) The questions in this part carry 50% of the total marks so you should spend about $1\frac{1}{2}$ hours in answering them.

(iii) You may answer the questions in any order. Write your answers in the answer book(s) provided, beginning each question on a new page.

(iv) Show all your working.

(v) To help the examiners, please write the numbers of the questions which you have attempted in this part at the foot of the front cover of the relevant answer book.

Question 1

For the complex number $\dfrac{1}{(1-i)^2}$, write down

(*a*) its real part,

(*b*) its imaginary part,

(*c*) its conjugate,

(*d*) its polar form in terms of the principal value of its argument. *(4 marks)*

Question 2

(*a*) Draw a rough sketch of the following set of points of the complex plane: $\{z : 1 < |z + 2i| \leqslant 2\}$.

(*b*) State whether or not this set is

(i) bounded,

(ii) a compact set,

(iii) a region.

(No explanation is required.) *(5 marks)*

Question 3

(*a*) Write down the definition of the function Log, the principal branch of the logarithm.

(*b*) For the function $z \to \dfrac{1}{(\text{Log}\, z - i\pi/2)^2}$, determine

(i) its domain,

(ii) the region on which it is analytic,

(iii) its derivative. *(7 marks)*

Question 4

State the Fundamental Theorem for contour integrals. Use it to evaluate the integral

$$\int_C \pi \cos(\pi z) dz\,,$$

where C is any contour from $1 - i$ to $1 + i$. *(5 marks)*

Question 13

(*a*) Let C be a circle $\{z : |z| = r\}$, where $r > 0$, with inside D, and let the function f be analytic on some region R containing $D \cup C$.

Prove that, for any point a in D,

$$f(a) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z-a}\,dz.$$

(You should *state*, without proof, the preliminary result about contour integrals that you require.) *(8 marks)*

(*b*) Let $M(r) = \max\{|f(z)| : |z| = r\}$.

Using part (*a*), prove that

$$|f(a) - f(0)| \leqslant \frac{|a|M(r)}{r - |a|} \qquad (|a| < r).$$ *(4 marks)*

(*c*) State Liouville's Theorem, and deduce it from the result of part (*b*). *(3½ marks)*

(*d*) Show that, if f is an entire function and $|f(z) - w_0| \geqslant 1$ for some fixed point w_0 in $\mathbb{C}$ and all z in $\mathbb{C}$, then f is a constant function. *(1½ marks)*

Question 14

(*a*) (i) Prove that, if $\sum_{n=0}^{\infty} a_n z_0^n$ converges and $z_0 \neq 0$, then $\sum_{n=0}^{\infty} a_n z^n$ converges absolutely for all z with $|z| < |z_0|$.

(ii) By means of a specific example, show that the series in part (i) may *not* converge at any point in $\{z : |z| > |z_0|\}$. (You need *not* prove your assertions.) *(4 marks)*

(*b*) Determine the first three non-vanishing coefficients in the Taylor Series of the function

$$f(z) = \frac{1+z}{(1-z)\cos z}$$

at $z = 0$. *(3 marks)*

(*c*) Determine the first three non-vanishing coefficients and the general coefficient in the Taylor Series of the function

$$g(z) = \frac{1}{z^2}$$

at $z = -1$. *State* its radius of convergence. *(4 marks)*

(*d*) (i) Find constants A and B such that

$$\frac{1}{(1+z)(2-z)} = \frac{A}{1+z} + \frac{B}{2-z}.$$

(ii) Hence, or otherwise, determine the first three non-vanishing coefficients and the general coefficient in the Taylor series of the function

$$h(z) = \frac{1}{2+z-z^2}$$

at $z = 1$. *State* its radius of convergence. *(6 marks)*

Question 15

(*a*) State and prove the Local Maximum Principle.

(You may assume that if $\int_a^b g = 0$, then $g = 0$ on $[a, b]$, where g is a real function continuous on $[a, b]$ such that $g \geqslant 0$ on $[a, b]$.) *(11 marks)*

(*b*) State the Maximum Principle and hence find the maximum value of the function $z \to \frac{z+2}{z-2}$ on the closed disc $\{z : |z| \leqslant 1\}$]. *(6 marks)*

TURN OVER

PART II

Instructions

(i) You should attempt not more than **TWO** questions from this part of the examination.

(ii) Each question in this part carries 17% of the total mark and so you should spend about 1 hour in answering this part.

(iii) You may answer the questions in any order. Write your answers in the answer book(s) provided, beginning each question on a new page.

(iv) Show all your working.

(v) To help the examiners, please write the numbers of the questions which you have attempted in this part at the foot of the front cover of the relevant answer book.

Question 11

(*a*) *Define* the complex hyperbolic functions *cosh* and *sinh*.
Use these definitions to show that

(i) $\cosh(z) = \cos(iz)$, and

(ii) $\sinh(z) = -i\sin(iz)$.

Hence, or otherwise, prove that

(iii) $\sinh(x + iy) = \sinh x \cos y + i \cosh x \sin y$. *(5 marks)*

(*b*) Sketch the graphs of

$$x \to \sinh x \quad \text{and} \quad x \to \cosh x,$$

for real x. (Great detail is *not* expected.) *(2 marks)*

(*c*) Determine the images under the mapping $z \to \sinh z$ of

(i) the coordinate axes, and

(ii) the lines $y = \pm \frac{\pi}{2}$ (where $z = x + iy$).

Illustrate your answer by means of diagrams; and state, for each of the four lines in (i) and (ii), whether the mapping is one-one. *(10 marks)*

Question 12

(*a*) Write down the Cauchy–Riemann equations for a function f differentiable at a point z_0. *(2 marks)*

(*b*) Prove that, if f is differentiable at a point z_0, then the Cauchy–Riemann equations are satisfied at the point z_0. *(9 marks)*

(*c*) Determine whether or not the Cauchy–Riemann equations hold for the function

$$f: x + iy \to x^2 + y^2 - y + ix$$

(i) at the point (0, 0);

(ii) at the point (1, 1). *(6 marks)*

Question 13

(*a*) Let C be a circle $\{z : |z| = r\}$, where $r > 0$, with inside D, and let the function f be analytic on some region R containing $D \cup C$.

Prove that, for any point a in D,

$$f(a) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z-a}dz\,.$$

(You should *state*, without proof, the preliminary result about contour integrals that you require.) *(8 marks)*

(*b*) Let $M(r) = \max\{|f(z)| : |z| = r\}$.

Using part (*a*), prove that

$$|f(a) - f(0)| \leqslant \frac{|a|M(r)}{r - |a|} \qquad (|a| < r)\,.$$ *(4 marks)*

(*c*) State Liouville's Theorem, and deduce it from the result of part (*b*). *(3½ marks)*

(*d*) Show that, if f is an entire function and $|f(z) - w_0| \geqslant 1$ for some fixed point w_0 in $\mathbb{C}$ and all z in $\mathbb{C}$, then f is a constant function. *(1½ marks)*

Question 14

(*a*) (i) Prove that, if $\sum_{n=0}^{\infty} a_n z_0^n$ converges and $z_0 \neq 0$, then $\sum_{n=0}^{\infty} a_n z^n$ converges absolutely for all z with $|z| < |z_0|$.

(ii) By means of a specific example, show that the series in part (i) may *not* converge at any point in $\{z : |z| > |z_0|\}$. (You need *not* prove your assertions.) *(4 marks)*

(*b*) Determine the first three non-vanishing coefficients in the Taylor Series of the function

$$f(z) = \frac{1+z}{(1-z)\cos z}$$

at $z = 0$. *(3 marks)*

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at $z = -1$. *State* its radius of convergence. *(4 marks)*

(*d*) (i) Find constants A and B such that

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(ii) Hence, or otherwise, determine the first three non-vanishing coefficients and the general coefficient in the Taylor series of the function

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at $z = 1$. *State* its radius of convergence. *(6 marks)*

Question 15

(*a*) State and prove the Local Maximum Principle.

(You may assume that if $\int_a^b g = 0$, then $g = 0$ on $[a, b]$, where g is a real function continuous on $[a, b]$ such that $g \geqslant 0$ on $[a, b]$.) *(11 marks)*

(*b*) State the Maximum Principle and hence find the maximum value of the function $z \to \frac{z+2}{z-2}$ on the closed disc $\{z : |z| \leqslant 1\}$. *(6 marks)*

TURN OVER

PART II

Instructions

(i) You should attempt not more than **TWO** questions from this part of the examination.

(ii) Each question in this part carries 17% of the total mark and so you should spend about 1 hour in answering this part.

(iii) You may answer the questions in any order. Write your answers in the answer book(s) provided, beginning each question on a new page.

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(ii) $\sinh(z) = -i\sin(iz)$.

Hence, or otherwise, prove that

(iii) $\sinh(x + iy) = \sinh x \cos y + i \cosh x \sin y$. *(5 marks)*

(*b*) Sketch the graphs of

$$x \to \sinh x \quad \text{and} \quad x \to \cosh x,$$

for real x. (Great detail is *not* expected.) *(2 marks)*

(*c*) Determine the images under the mapping $z \to \sinh z$ of

(i) the coordinate axes, and

(ii) the lines $y = \pm \frac{\pi}{2}$ (where $z = x + iy$).

Illustrate your answer by means of diagrams; and state, for each of the four lines in (i) and (ii), whether the mapping is one-one. *(10 marks)*

Question 12

(*a*) Write down the Cauchy–Riemann equations for a function f differentiable at a point z_0. *(2 marks)*

(*b*) Prove that, if f is differentiable at a point z_0, then the Cauchy–Riemann equations are satisfied at the point z_0. *(9 marks)*

(*c*) Determine whether or not the Cauchy–Riemann equations hold for the function

$$f: x + iy \to x^2 + y^2 - y + ix$$

(i) at the point (0, 0);

(ii) at the point (1, 1). *(6 marks)*

PART III

Instructions

(i) You should attempt **ONE** question from this part of the examination.

(ii) Each question in this part carries 16% of the total mark and so you should spend about $\frac{1}{2}$ hour in answering this part.

(iii) Write your answer in the answer book(s) provided and show all your working.

(iv) To help the examiners, please write the number of the question which you have attempted in this part at the foot of the front cover of the relevant answer book.

Question 17

(*a*) If f is analytic on a neighbourhood of α and $f'(\alpha) \neq 0$, prove that, for any pair of smooth arcs γ_1 and γ_2 through α, $f \circ \gamma_1$ and $f \circ \gamma_2$ are smooth arcs through $f(\alpha)$ such that the angle between them at $f(\alpha)$ is the same as the angle between γ_1 and γ_2 at α. *(5 marks)*

(*b*) Define the terms *bilinear transformation, conformal transformation.* *(1½ marks)*

(*c*) Let $C_1 = \{z : |z - i| = 1\}$, $C_2 = \{z : |z - 1| = 1\}$; and let D_1, D_2 be the insides of the circles C_1, C_2 respectively. Let $R_1 = D_1 \cap D_2$.

(i) Sketch C_1, C_2 and R_1.
C_1 and C_2 intersect at the origin; at what angle do they intersect?

(ii) Prove that the image of R_1 under the bilinear transformation

$$B : z \to \frac{z}{z - (1 + i)}$$

is the quarter-plane $R_2 = \left\{z : \frac{3\pi}{4} < \arg z < \frac{5\pi}{4}\right\}$.

(iii) Determine an analytic equivalence of the quarter-plane R_2 with the right half-plane $R_3 = \{z : \mathrm{Re}\, z > 0\}$.

(iv) Determine a bilinear transformation that maps R_3 onto the unit disc $R_4 = \{z : |z| < 1\}$.

(v) Using the results of parts (ii), (iii) and (iv), determine an analytic equivalence of R_1 with R_4. *(9½ marks)*

Question 18

(*a*) Write down the definitions of the Laplace transform F of a function f and the corresponding inversion formula, clearly stating a set of conditions under which this inversion formula is valid. *(6 marks)*

(*b*) Let f be a real function that satisfies the differential equation

$$f''(t) + 2f'(t) - 3f(t) = t, \quad t > 0,$$

and let $f(0) = 0$, $f'(0) = 1$. Determine the Laplace transform of f. *(3 marks)*

(*c*) Show that you may use the inversion formula to evaluate the function f of part (*b*) and hence find f. *(7 marks)*

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M332

Solutions and Mark Scheme of the Specimen Examination Paper

PART I

Question 1

$(1-i)^2 = 1 - 2i + i^2 = -2i$ and so $z = \dfrac{1}{(1-i)^2} = -\dfrac{1}{2i} = \dfrac{i}{2}$ — 1M

(a) $\operatorname{Re} z = 0$ — $\frac{1}{2}$M

(b) $\operatorname{Im} z = \frac{1}{2}$ — $\frac{1}{2}$M

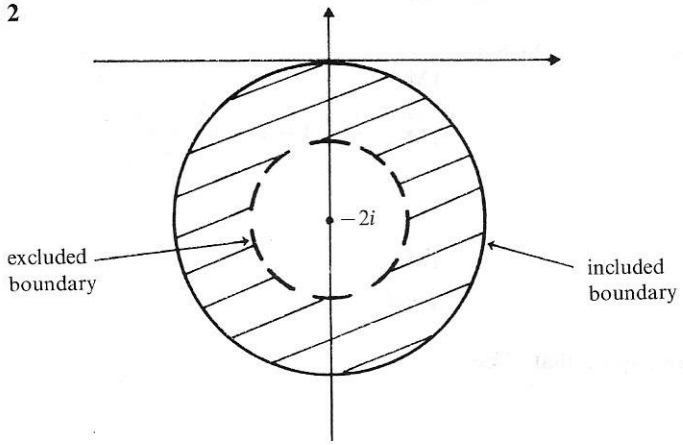
(c) $\bar{z} = -\dfrac{i}{2}$ — $\frac{1}{2}$M

(d) $z = \frac{1}{2}(\cos \pi/2 + i \sin \pi/2)$ — $\frac{1}{2}$M for correct modulus; $\frac{1}{2}$M for Arg; $\frac{1}{2}$M for correct form

Total 4

Question 2

(a)



2M ($\frac{1}{2}$M for showing that larger circle is included, $\frac{1}{2}$M for showing that smaller circle is excluded)

(b) (i) The set is bounded. — 1M

(ii) It is not compact. — 1M

(iii) It is not a region. — 1M

Total 5

Comment: Note that each of (i)–(iii) is answered explicitly. Reasons for the answers given are:

(i) the set is bounded since it is contained in the closed disc $\{z: |z| \leq 5\}$;

(ii) it is not compact since it is not closed;

(iii) it is not a region since it is not open.

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Question 3

(a) Log is the inverse function of the exponential function with domain restricted to $\{z: -\pi < \text{Im } z < \pi\}$. The domain of Log is

$$D_L = \{z: z \text{ is not zero or a negative real number}\}.$$ 2M

Comment: Any other equivalent statement would be accepted provided that the domain of Log was correctly stated.

(b) (i) Let $f(z) = \dfrac{1}{(\text{Log } z - i\pi/2)^2}$. The domain of f is the set of all $z \in \mathbb{C}$ for which Log z is defined and Log $z \neq i\pi/2$. Log $z = i\pi/2$ implies that

$$z = \exp(i\pi/2)$$
$$= i,$$

1M for $z \neq i$

and so the domain of f is $D_L - \{i\}$, where D_L is defined above. 1M for the domain

(ii) f is analytic on its domain, which is a region. Hence its domain is the required region. 1M

(iii) $f'(z) = \dfrac{-2}{(\text{Log } z - i\pi/2)^3} \cdot \dfrac{1}{z}$. 2M for accurate answer

Total 7

Question 4

(a) Let f be analytic on a region R, and Γ be a contour in R. Suppose also that f' is continuous on R. Then

$$\int_\Gamma f'(z)dz = f(\beta) - f(\alpha),$$ 1M

where α, β are the initial, final points of Γ. Furthermore, if Γ is closed, then

$$\int_\Gamma f'(z)dz = 0.$$ 1M

(b) $\pi \cos(\pi z)$ is continuous on $\mathbb{C}$; and is the derivative of $f(z) = \sin(\pi z)$, which is entire. So the Fundamental Theorem applies, and 1M

$$\int_\Gamma \pi \cos(\pi z)dz = \Big[\sin(\pi z)\Big]_{1-i}^{1+i}$$ 1M
$$= \sin \pi(1 + i) - \sin \pi(1 - i)$$
$$= 2 \cos \pi \sin(i\pi)$$
$$= -2i \sinh \pi.$$ 1M

Total 5

Comment: Remember that in the statement of the theorem we require that f' be continuous on R.

Question 5

(a) Let f be analytic on a region containing the circle $C = \{z: |z - \alpha| = r\}$ and its inside D. Then for any z in D,

$$f'(z) = \frac{1}{2\pi i} \int_C \frac{f(w)}{(w - z)^2} dw.$$ 2M (1M for conditions. 1M for the formula)

(b) The function sin is entire and so satisfies the hypotheses of the theorem. Putting $f = \sin$ and $z = 0$ we have

$$\cos(0) = \frac{1}{2\pi i} \int_C \frac{\sin w}{w^2} dw$$ 1M ($\frac{1}{2}$M for checking conditions, $\frac{1}{2}$M for the equation)

so that

$$\int_C \frac{\sin w}{w^2} dw = 2\pi i.$$ 1M for accurate answer

Total 4

Comment: In (a) you could have quoted the result for simply-connected regions.

Question 6

$$\frac{1}{(1-z)^2} = \frac{1}{(2-(z+1))^2}$$

$$= \frac{1}{4\left(1-\frac{(z+1)}{2}\right)^2}$$

$$= \frac{1}{4}\left(1+2\left(\frac{z+1}{2}\right)+3\left(\frac{z+1}{2}\right)^2+\cdots\right), \quad \text{provided that } \left|\frac{z+1}{2}\right|<1,$$

1M for correct expansion

$$= \frac{1}{4}+\frac{2(z+1)}{2^3}+\frac{3(z+1)^2}{2^4}+\cdots$$

$$= \sum_{n=0}^{\infty}\frac{(n+1)}{2^{n+2}}(z+1)^n.$$

2M for correct series (it does not have to be in this form)

Since the expression is valid for $|z+1|<2$, the Taylor series has radius of convergence 2.

1M for reasoning

Total 4

Comment: Should you not know the expansion for $(1-z)^{-2}$ you could calculate the Taylor coefficients directly by evaluating $f(-1), f'(-1), f''(-1), \ldots, f^{(n)}(-1), \ldots$. An alternative argument for working out its radius of convergence is as follows: the function $f: z \longrightarrow \frac{1}{(1-z)^2}$ is analytic on $\mathbb{C}-\{1\}$ and so the series is convergent provided $|z+1|<2$.

Question 7

(a) (i) $$f(z) = \frac{e^z}{(z-1)^2}$$

$$= \frac{e\cdot e^{z-1}}{(z-1)^2}$$

$$= \frac{e}{(z-1)^2}\cdot\left(1+(z-1)+\frac{(z-1)^2}{2!}+\cdots\right)$$

$$= \frac{e}{(z-1)^2}+\frac{e}{(z-1)}+\frac{e}{2!}+\frac{e(z-1)}{3!}+\cdots.$$

2M for accurate expansion

(ii) $$g(z) = z\cos\frac{1}{z}$$

$$= z\left(1-\frac{1}{2!z^2}+\frac{1}{4!z^4}-\frac{1}{6!z^6}+\cdots\right)$$

$$= z-\frac{1}{2!z}+\frac{1}{4!z^3}-\frac{1}{6!z^5}\cdots.$$

2M for accurate expansion

(b) (i) $z=1$ is a pole of f of order 2. 1M

(ii) $z=0$ is an essential singularity of g. 1M

Total 6

Question 8

Let R be a simply-connected region in $\mathbb{C}$, $\alpha_1, \ldots, \alpha_n$ be points of R, and f a function analytic on R except for singularities at $\alpha_1, \ldots, \alpha_n$. Let Γ be any simply-closed contour in R not passing through any of $\alpha_1, \ldots, \alpha_n$. Then

$$\int_\Gamma f(z)dz = 2\pi i \sum \mathrm{Res}(f, \alpha_j),$$

where the sum is taken over those singularities that lie inside Γ.

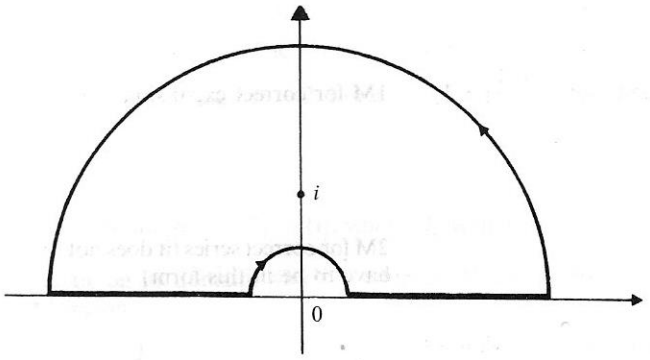
3M

Total 3

Question 9

(a) Let $f: z \longrightarrow \dfrac{1-z}{z(z^2+1)}$. To evaluate the integral, integrate f around the contour shown.

1M for integrand



1M for contour

The poles relevant to the integral are at $z = 0$ and $z = i$.

1M for relevant poles ($\frac{1}{2}$M each)

(b) $\text{Res}(f, 0) = \lim\limits_{z \to 0} \dfrac{1-z}{z^2+1} = 1,$

2M

$\text{Res}(f, i) = \lim\limits_{z \to i} \dfrac{1-z}{z(z+i)} = \dfrac{1-i}{i(2i)} = -\dfrac{1}{2} + \dfrac{i}{2}.$

2M

Total 7

Question 10

(a) Let $\{f_n\}$ be a sequence of functions defined on a set A, and suppose that $\{M_n\}$ is a sequence of real numbers such that $|f_n(z)| \leq M_n$ for all z in A. Suppose, moreover, that $\sum\limits_{n=1}^{\infty} M_n$ converges.

Then, for each z in A, the series $\sum\limits_{n=1}^{\infty} f_n(z)$ converges (in fact, it converges absolutely), and $\sum\limits_{n=1}^{\infty} f_n$ converges uniformly on A to the function

$$z \longrightarrow \sum_{n=1}^{\infty} f_n(z).$$

3M

(b) $|n^{-z}| = |e^{-z \operatorname{Log} n}|$

$= |e^{-\operatorname{Re} z \operatorname{Log} n} e^{-i \operatorname{Im} z \operatorname{Log} n}|$

$= |e^{-\operatorname{Re} z \operatorname{Log} n}|$

$= n^{-\operatorname{Re} z}.$

1M

Now $\operatorname{Re} z > 2$, and so

$|n^{-z}| = n^{-\operatorname{Re} z}$

$< n^{-2}.$

$\frac{1}{2}$M

$\sum\limits_{n=1}^{\infty} \dfrac{1}{n^2}$ converges, and so, by the M-test $\sum\limits_{n=1}^{\infty} \dfrac{1}{n^z}$ converges uniformly on $\{z: \operatorname{Re} z > 2\}$.

$\frac{1}{2}$M for conclusion

Total 5

Part I is marked out of a total of 50.

PART II

Question 11

(a) $\cosh z = \frac{1}{2}(e^z + e^{-z})$, $\sinh z = \frac{1}{2}(e^z - e^{-z})$. 2M

(i) $\cos(iz) = \frac{1}{2}(e^{i(iz)} + e^{-i(iz)})$

$= \frac{1}{2}(e^{-z} + e^z) = \cosh z.$ 1M

(ii) $-i\sin(iz) = -i\dfrac{1}{2i}(e^{i(iz)} - e^{-i(iz)})$

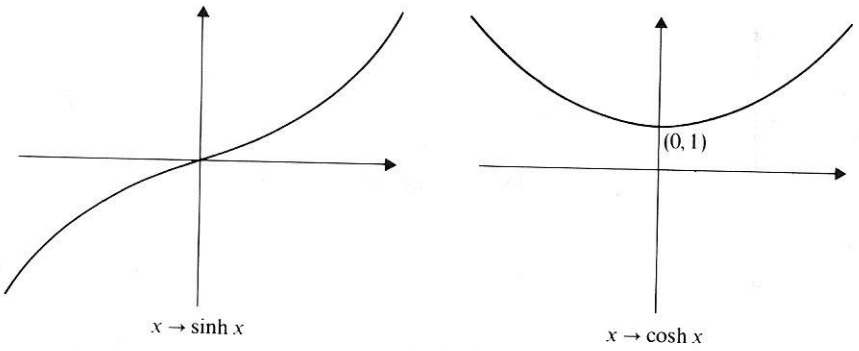
$= -\dfrac{1}{2}(e^{-z} - e^z) = \sinh z.$ 1M

(iii) $\sinh(x + iy) = -i\sin(ix - y)$

$= -i[\sin(ix)\cos y - \cos(ix)\sin y]$

$= \sinh x \cos y + i\cosh x \sin y$ 1M

(b)



2M

Comment: Only the general shape is required, but the point (0, 1) should be indicated on the second graph.

(c) Putting $u + iv = \sinh(x + iy)$

$= \sinh x \cos y + i\cosh x \sin y,$

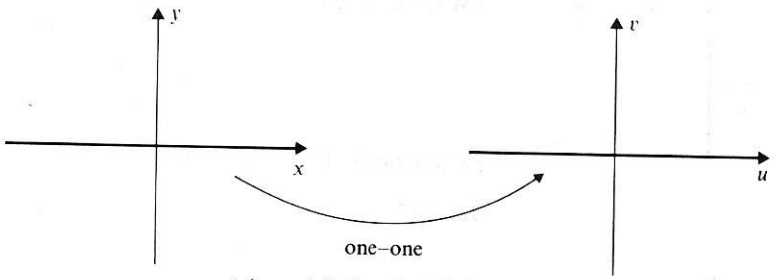
and equating real and imaginary parts gives

(*) $u = \sinh x \cos y$, $v = \cosh x \sin y$. $\frac{1}{2}$M

(i) To determine the image of a typical point on the x-axis, put $y = 0$ in (*). Then its image is

$u = \sinh x, \quad v = 0.$ 1M

As x varies from $-\infty$ to $+\infty$, so u varies from $-\infty$ to $+\infty$; hence the image of the x-axis is the whole u-axis. 1M



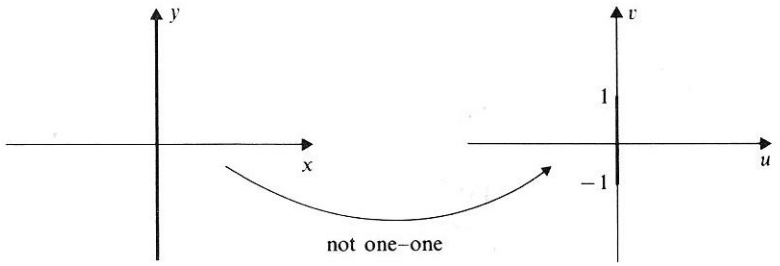
$\frac{1}{2}$M (for diagram)

(since sinh is a 1–1 function) $\frac{1}{2}$M for (1–1)

To determine the image of a typical point on the y-axis, put $x = 0$ in (*). Then its image is

$u = 0, \quad v = \sin y.$ $\frac{1}{2}$M

As y varies from $-\infty$ to $+\infty$, v varies between ± 1. So the image of the y-axis is the segment $[-1, 1]$ of the v-axis. 1M



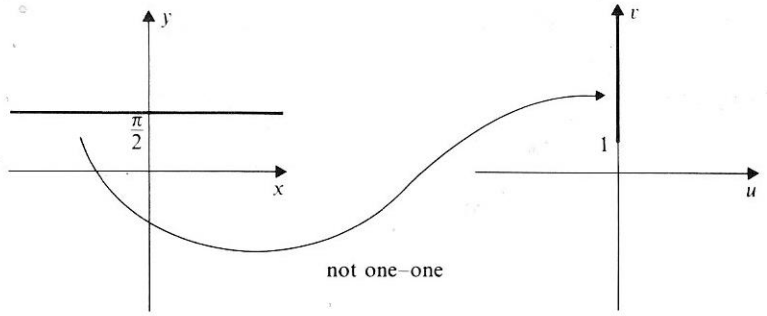
½M for diagram

(since sin is not 1–1) ½M for 1–1

(ii) Now let $y = \frac{\pi}{2}$ in (∗); this gives

$$u = 0, \quad v = \cosh x.$$

As x varies between $-\infty$ and $+\infty$, v varies between 1 and ∞; so the image of $y = \frac{\pi}{2}$ is that part of the v-axis $[1, \infty)$. 1M



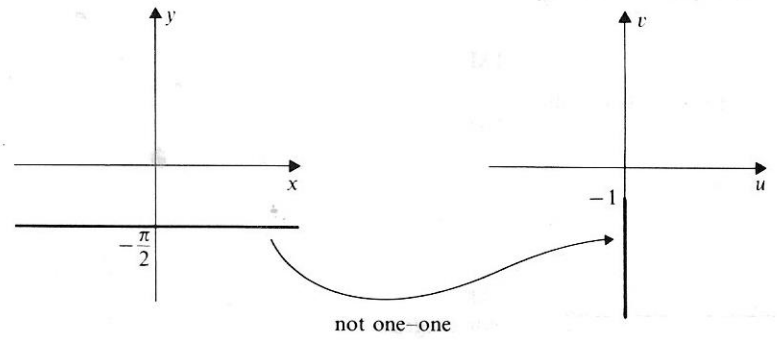
½M for diagram

(since cosh is not 1–1) ½M for not 1–1

Now let $y = -\frac{\pi}{2}$ in (∗); this gives

$$u = 0, \quad v = -\cosh x.$$

As x varies between $-\infty$ and $+\infty$, v varies between -1 and $-\infty$; so the image of $y = -\frac{\pi}{2}$ is that part of the v-axis $(-\infty, -1]$. 1M



½M for diagram

½M for not 1–1

(since $-\cosh$ is not 1–1)

Total 17

Question 12

(a) If $z_0 = x_0 + iy_0$,

$$\frac{\partial u}{\partial x}(x_0, y_0) = \frac{\partial v}{\partial y}(x_0, y_0) \text{ and } \frac{\partial u}{\partial y}(x_0, y_0) = -\frac{\partial v}{\partial x}(x_0, y_0). \quad \text{2M}$$

(b) Since f is differentiable at z_0, given any $\varepsilon > 0$ there is some $\delta > 0$ such that

$$\left| \frac{f(z) - f(z_0)}{z - z_0} - f'(z_0) \right| < \varepsilon$$

whenever $0 < |z - z_0| < \delta$. 1M

Hence, if α is real and $0 < |\alpha| < \delta$,

$$\left|\frac{f(z_0+\alpha)-f(z_0)}{\alpha} - f'(z_0)\right| < \varepsilon; \qquad \tfrac{1}{2}\text{M}$$

that is

$$\left|\frac{u(x_0+\alpha,y_0)-u(x_0,y_0)}{\alpha} + i\frac{v(x_0+\alpha,y_0)-v(x_0,y_0)}{\alpha} - f'(z_0)\right| < \varepsilon. \qquad 1\text{M}$$

Since $|\text{Re}\,\xi| \leq |\xi|$ and $|\text{Im}\,\xi| \leq |\xi|$ 1M

for any $\xi \in \mathbb{C}$, it follows that whenever $0 < |\alpha| < \delta$,

$$\left|\frac{u(x_0+\alpha,y_0)-u(x_0,y_0)}{\alpha} - \text{Re}\,f'(z_0)\right| < \varepsilon \qquad \tfrac{1}{2}\text{M}$$

and

$$\left|\frac{v(x_0+\alpha,y_0)-v(x_0,y_0)}{\alpha} - \text{Im}\,f'(z_0)\right| < \varepsilon. \qquad \tfrac{1}{2}\text{M}$$

Letting $\alpha \longrightarrow 0$ we get that

$$\frac{\partial u}{\partial x}(x_0,y_0) = \text{Re}f'(z_0), \quad \frac{\partial v}{\partial x}(x_0,y_0) = \text{Im}f'(z_0). \qquad 1\text{M}$$

On the other hand, if β is real and $0 < |\beta| < \delta$,

$$\left|\frac{f(z_0+i\beta)-f(z_0)}{i\beta} - f'(z_0)\right| < \varepsilon; \qquad \tfrac{1}{2}\text{M}$$

that is,

$$\left|\frac{u(x_0,y_0+\beta)-u(x_0,y_0)}{i\beta} + i\frac{v(x_0,y_0+\beta)-v(x_0,y_0)}{i\beta} - f'(z_0)\right| < \varepsilon. \qquad \tfrac{1}{2}\text{M}$$

Hence whenever $0 < |\beta| < \delta$,

$$\left|-\frac{u(x_0,y_0+\beta)-u(x_0,y_0)}{\beta} - \text{Im}\,f'(z_0)\right| < \varepsilon$$

and

$$\left|\frac{v(x_0,y_0+\beta)-v(x_0,y_0)}{\beta} - \text{Re}\,f'(z_0)\right| < \varepsilon. \qquad 1\tfrac{1}{2}\text{M}$$

Letting $\beta \longrightarrow 0$ gives that

$$\frac{\partial u}{\partial y}(x_0,y_0) = -\text{Im}f'(z_0), \quad \frac{\partial v}{\partial y}(x_0,y_0) = \text{Re}f'(z_0). \qquad \tfrac{1}{2}\text{M}$$

Combining these two conclusions gives

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \text{ at } (x_0,y_0). \qquad \tfrac{1}{2}\text{M}$$

(c) Here $u = x^2 + y^2 - y$ and $v = x$, so that

$$\left.\begin{aligned}\frac{\partial u}{\partial x} &= 2x\\ \frac{\partial u}{\partial y} &= 2y-1\end{aligned}\right\} \quad \text{and} \quad \left.\begin{aligned}\frac{\partial v}{\partial x} &= 1\\ \frac{\partial v}{\partial y} &= 0\end{aligned}\right\} \qquad 2\text{M}$$

(i) Hence at (0, 0)

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = 0, \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} = 1.$$

So the Cauchy–Riemann equations hold at (0, 0). 2M

(ii) And at (1, 1),

$$\frac{\partial u}{\partial x} = 2, \quad \frac{\partial u}{\partial y} = \frac{\partial v}{\partial x} = 1, \quad \frac{\partial v}{\partial y} = 0;$$

so the Cauchy–Riemann equations do not hold at (1, 1). 2M

Total 17

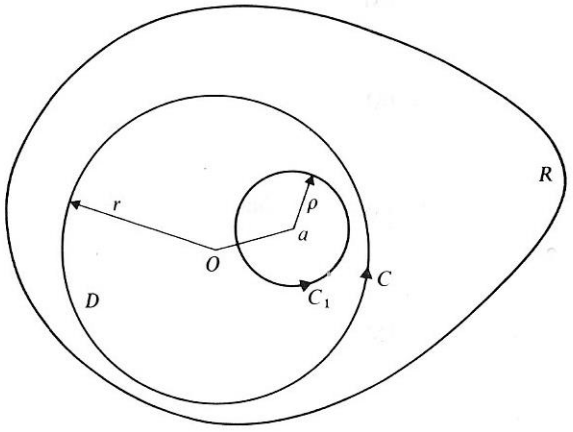
Question 13

(a) *Preliminary Result*

If C_1 is any circle $\{z: |z-a| = \rho\}$ where

$0 < \rho < r - |a|$, then

(*) $\displaystyle\int_{C_1} \frac{f(z)}{z-a}\,dz = \int_C \frac{f(z)}{z-a}\,dz.$ 1M



Proof

We know that

(**) $\displaystyle\int_{C_1} \frac{dz}{z-a} = 2\pi i.$ ½M

So let

$$I = \int_C \frac{f(z)}{z-a}\,dz - 2\pi i f(a);$$

we will show that $I = 0$. 1M

Using the results (*) and (**),

$$\begin{aligned} I &= \int_C \frac{f(z)}{z-a}\,dz - f(a)\int_{C_1} \frac{dz}{z-a} \\ &= \int_{C_1} \frac{f(z)}{z-a}\,dz - f(a)\int_{C_1} \frac{dz}{z-a} \\ &= \int_{C_1} \frac{f(z)-f(a)}{z-a}\,dz. \end{aligned}$$ 1M

We now make this integral arbitrarily small by choosing p. ½M

Now

$$\begin{aligned} \left|\int_{C_1} \frac{f(z)-f(a)}{z-a}\,dz\right| &\le \int_{C_1} \left|\frac{f(z)-f(a)}{z-a}\right| |dz| \\ &= \frac{1}{\rho}\int_{C_1} |f(z)-f(a)|\,|dz|. \end{aligned}$$ 1M

Next choose any $\varepsilon > 0$. Since f is continuous at a, there exists some $\delta > 0$ such that

$|f(z) - f(a)| < \varepsilon$ whenever $|z-a| < \delta$. 1M

So if we choose $\rho < \delta$, then

$$\begin{aligned} \frac{1}{\rho}\int_{C_1} |f(z)-f(a)| \cdot |dz| &\le \frac{1}{\rho} \cdot \varepsilon \cdot 2\pi\rho \\ &= 2\pi\varepsilon, \end{aligned}$$

Hence

$|I| \le 2\pi\varepsilon$. 1M

Since this holds for all $\varepsilon > 0$, $I = 0$; thus

$$\int_C \frac{f(z)}{z-a}\,dz = 2\pi i f(a).$$ 1M

(b) Part (a) with $a = 0$ gives

$$f(0) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z}dz. \qquad \text{1M}$$

Subtracting this from the formula in part (a) for $f(a)$ gives

$$f(a) - f(0) = \frac{1}{2\pi i}\int_C f(z)\left\{\frac{1}{z-a} - \frac{1}{z}\right\}dz$$

$$= \frac{1}{2\pi i}\int_C \frac{f(z)\cdot a}{z(z-a)}dz. \qquad \text{1M}$$

Using the Estimation Theorem,

$$|f(a) - f(0)| \le \frac{1}{2\pi}\max_{z\in C}\left\{\frac{|f(z)|\cdot|a|}{|z|\cdot|z-a|}\right\}\cdot l(C)$$

$$\le \frac{1}{2\pi}\cdot\frac{M(r)|a|}{r(r-|a|)}\cdot 2\pi r$$

$$= \frac{|a|M(r)}{r-|a|}, \qquad \text{1M}$$

since on C, by the Cunning Form of the Triangle Inequality,

$$|z-a| \ge ||z| - |a||$$

$$= r - |a|, \qquad \text{as } |a| < r. \qquad \text{1M}$$

(c) *Liouville's Theorem:* If f is a bounded entire function, it is necessarily constant. 1M

Proof Suppose $|f(z)| \le M$ for all $z\in\mathbb{C}$.

Then, from part (b), for any $a\in\mathbb{C}$ we have that

$$|f(a) - f(0)| \le \frac{M|a|}{r-|a|}. \qquad \text{1M}$$

Letting $r \longrightarrow \infty$ gives that

$$|f(a) - f(0)| \le 0,$$

so that

$$f(a) = f(0). \qquad \text{1M}$$

Since this holds for all $a\in\mathbb{C}$, f is necessarily constant. $\frac{1}{2}$M

(d) Since $|f(z) - w_0| \ge 1$ for all $z\in\mathbb{C}$, the function $\dfrac{1}{f(z)-w_0}$ is an entire function bounded by 1. Hence $\dfrac{1}{f(z)-w_0}$ is constant, and so f too is constant. $1\frac{1}{2}$M

Total 17

Question 14

(a) (i) Since $\sum_{n=0}^{\infty} a_n z_0^n$ converges,

$$\lim_{n\to\infty} |a_n z_0^n| = 0.$$

In particular, the sequence $\{|a_n z_0^n|\}_{n=0}^{\infty}$ must be bounded; that is, for some K,

$$|a_n z_0^n| \le K \text{ for all } n. \qquad \text{1M}$$

Then

$$|a_n z^n| = |a_n z_0^n|\left|\frac{z}{z_0}\right|^n$$

$$\le K\left|\frac{z}{z_0}\right|^n. \qquad \frac{1}{2}\text{M}$$

When $|z/z_0| < 1$, the series $\sum_{n=0}^{\infty} K\left|\frac{z}{z_0}\right|^n$ converges. Hence by the Comparison Test, so does $\sum_{n=0}^{\infty} |a_n z^n|$. 1M

Thus $\sum_{n=0}^{\infty} a_n z^n$ is absolutely convergent, when $|z| < |z_0|$. $\frac{1}{2}$M

(ii) The power series $\sum_{n=0}^{\infty} \frac{1}{n^2} z^n$ has radius of convergence 1, and converges at $z_0 = 1$. It does *not* converge in $\{|z| > 1\}$. 1M

(b) Near 0,

$$\cos z = 1 - \frac{z^2}{2!} + \cdots; \qquad \text{1M}$$

hence, near 0,

$$\begin{aligned} f(z) &= \frac{1+z}{(1-z)(1-\frac{1}{2}z^2+\cdots)} \\ &= (1+z)(1-z)^{-1}(1-[\tfrac{1}{2}z^2+\cdots])^{-1} \\ &= (1+z)(1+z+z^2+\cdots)(1+\tfrac{1}{2}z^2+\cdots) \\ &= (1+2z+2z^2+\cdots)(1+\tfrac{1}{2}z^2+\cdots) \\ &= 1+2z+\tfrac{5}{2}z^2+\cdots . \end{aligned} \qquad \text{2M}$$

Comment: Always be careful that some of the terms included in the "dots" do not make a contribution to the coefficients you are looking for!

(c) Here

$$g(z) = z^{-2};$$

so that successive differentiation gives

$$g'(z) = (-2)z^{-3},$$

$$g''(z) = (-2)(-3)z^{-4}; \text{ and}$$

$$g^{(n)}(z) = (-2)(-3)\ldots(-(n+1))z^{-(n+2)} = (-1)^n(n+1)!z^{-(n+2)}.$$

Putting $z = -1$ gives

$$g(-1) = 1,$$

$$g'(-1) = 2,$$

$$g''(-1) = 6,$$

$$g^{(n)}(-1) = (n+1)! \qquad \text{2M}$$

Hence by Taylor's Theorem, near $z = -1$ we have that

$$\begin{aligned} g(z) &= g(-1) + \frac{g'(-1)}{1!}(z+1) + \frac{g''(-1)}{2!}(z+1)^2 + \cdots \\ &\qquad \cdots + \frac{g^{(n)}(-1)}{n!}(z+1)^n + \cdots \\ &= 1 + 2z + 3z^2 + \cdots + (n+1)(z+1)^n + \cdots. \end{aligned} \qquad \text{1}\tfrac{1}{2}\text{M}$$

Radius of convergence = 1. $\qquad \tfrac{1}{2}$M

(d) (i) Multiplying both sides by $(1+z)(2-z)$ gives that

$$1 = A(2-z) + B(1+z).$$

Substituting $z = 2$ gives $B = \frac{1}{3}$, and substituting $z = -1$ gives $A = \frac{1}{3}$. 1M

(ii) Now

$$h(z) = \frac{\frac{1}{3}}{1+z} + \frac{\frac{1}{3}}{2-z}.$$

First, near 1 we have that

$$\begin{aligned} \frac{1}{1+z} &= \frac{1}{2+(z-1)} \\ &= \frac{1}{2} \times \frac{1}{1+\frac{z-1}{2}} \\ &= \frac{1}{2}\left[1 - \frac{z-1}{2} + \left(\frac{z-1}{2}\right)^2 + \cdots + (-1)^n\left(\frac{z-1}{2}\right)^n + \cdots\right] \\ &= \frac{1}{2} - \frac{1}{4}(z-1) + \frac{1}{8}(z-1)^2 + \cdots + \frac{(-1)^n}{2^{n+1}}(z-1)^n + \cdots. \end{aligned} \qquad \text{2M}$$

Also, near 1 we have that

$$\begin{aligned} \frac{1}{2-z} &= \frac{1}{1-(z-1)} \\ &= 1 + (z-1) + (z-1)^2 + \cdots + (z-1)^n + \cdots. \end{aligned} \qquad \text{1M}$$

It follows, after addition, that

$$h(z) = \frac{1}{3}\left\{\frac{3}{2} + \frac{3}{4}(z-1) + \frac{9}{8}(z-1)^2 + \cdots + \left\{1 + \frac{(-1)^n}{2^{n+1}}\right\}(z-1)^n + \cdots\right\}$$

$$= \frac{1}{2} + \frac{1}{4}(z-1) + \frac{3}{8}(z-1)^2 + \cdots + \frac{1}{3}\left\{1 + \frac{(-1)^n}{2^{n+1}}\right\}(z-1)^n + \cdots \qquad \text{1M}$$

The radius of convergence is 1. 1M

Total 17

Question 15

(a) Let f be analytic on a region R. If f is not constant on R, then $|f|$ has no local maximum at any point in R. 2M for statement

Suppose that f is not constant on R, but that $|f|$ has a local maximum at a point α in R. Then, since R is open, there is $\varepsilon > 0$ such that 1M for method of proof

$$|f(z)| \leq |f(\alpha)| \quad \text{whenever} \quad |z - \alpha| < \varepsilon. \qquad (1)$$ 1M for open disc

From Gauss's Mean Value Theorem,

$$f(\alpha) = \frac{1}{2\pi}\int_0^{2\pi} f(\alpha + re^{i\theta})\, d\theta;$$ 1M for use of Gauss's Theorem

therefore

$$|f(\alpha)| = \frac{1}{2\pi}\left|\int_0^{2\pi} f(\alpha + re^{i\theta})\, d\theta\right|$$

$$\leq \frac{1}{2\pi}\int_0^{2\pi} |f(\alpha + re^{i\theta})|\, d\theta,$$ 1M for inequality

using the result that $\left|\int_a^b \phi\right| \leq \int_a^b |\phi|$,

$$\leq \frac{1}{2\pi}\int_0^{2\pi} |f(\alpha)|\, d\theta \qquad \text{by (1)}$$

$$= |f(\alpha)|,$$

and so $|f(\alpha)| = \frac{1}{2\pi}\int_0^{2\pi} |f(\alpha + re^{i\theta})|\, d\theta$, that is, 1M for equality

$$\int_0^{2\pi} (|f(\alpha)| - |f(\alpha + re^{i\theta})|)\, d\theta = 0.$$

Applying the result of real analysis to the non-negative function 1M for use of real analysis result

$$g(\theta) = |f(\alpha)| - |f(\alpha + re^{i\theta})|$$

it follows that $g(\theta) = 0$, that is

$$|f(\alpha)| = |f(\alpha + re^{i\theta})|, \quad \text{whenever } 0 \leq \theta \leq 2\pi.$$ 1M for equality

But this is true for any $r \in (0, \varepsilon)$ and so $|f|$ is constant on $D = \{z : |z - \alpha| < \varepsilon\}$. 1M for result on disc
Hence f is constant on D, and so, by the Uniqueness Theorem, constant on R. 1M for result on R
This is a contradiction.

(b) Let f be analytic on a bounded region R, and continuous on $\bar{R}$. Then $|f|$ has a maximum on $\bar{R}$ at a point on the boundary of R. — 2M for statement

The function $f: z \longrightarrow \dfrac{z+2}{z-2}$ is analytic on $D = \{z: |z| \leq 1\}$ and so has a maximum on the boundary of R, the circle $|z| = 1$. Now on $|z| = 1$ — 1M for checking conditions

$$|z + 2| \leq |z| + 2 = 3$$

and

$$|z - 2| \geq ||z| - 2| = |-1| = 1.$$

Hence on $|z| = 1$

$$|f(z)| \leq \frac{3}{1},$$ — 2M for accurate inequality

and so an upper bound for $|f|$ on $|z| = 1$ is 3. That it is *the* maximum follows since $|f(1)| = 3$. — 1M for this line

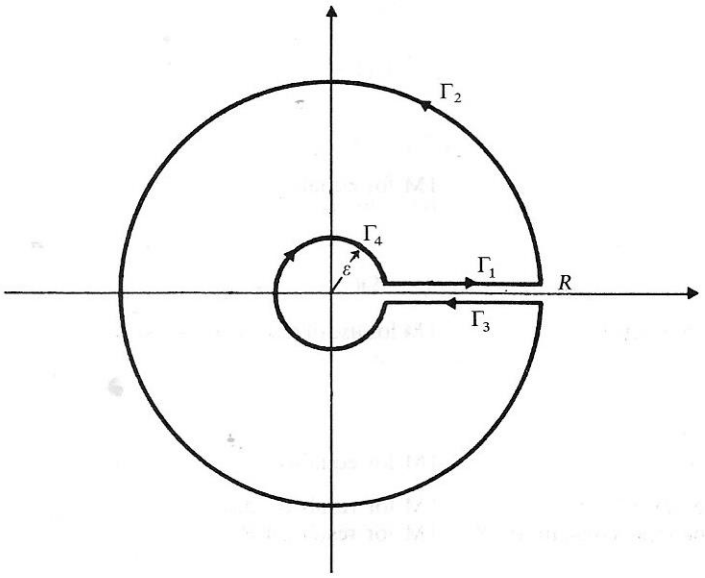
Total 17

Question 16

(a) (f_1, R_1) and (f_2, R_2) are said to be direct analytic continuations of each other if $f_1(z) = f_2(z)$ for all z in some non-empty subset of $R_1 \cap R_2$. — 2M

The sequence $\{(f_1, R_1), (f_2, R_2), \ldots, (f_n, R_n)\}$ is said to be a complete chain if (f_{k+1}, R_{k+1}) is the direct analytic continuation of (f_k, R_k) for $k = 1, 2, \ldots, n-1$, and $R_1 = R_n$. — 2M

(b) Consider the complete analytic function $f: z \longrightarrow \dfrac{z^{-1/2}}{1+z}$, which has a branch point at 0. Label the given contour as in the diagram below, where points of Γ_1 have argument 0 and points of Γ_3 have argument 2π.



The function f has a pole inside the contour at $e^{i\pi}$ with residue $(e^{i\pi})^{-1/2} = -i$. — 1M for pole
Applying the Residue Theorem, we obtain — 1M for residue

$$\int_\Gamma \frac{z^{-1/2}}{1+z} = \int_{\Gamma_1} f + \int_{\Gamma_2} f + \int_{\Gamma_3} f + \int_{\Gamma_4} f$$ — 1M for splitting up integral

$$= \int_\varepsilon^R \frac{dx}{(1+x)\sqrt{x}} + \int_0^{2\pi} \frac{(Re^{i\theta})^{-1/2} i Re^{i\theta}}{1 + Re^{i\theta}}\, d\theta$$

$$+ \int_R^\varepsilon \frac{(e^{2\pi i}x)^{-1/2}}{1+x}\, dx + \int_{2\pi}^0 \frac{(\varepsilon e^{i\theta})^{-1/2} i\varepsilon e^{i\theta}}{1 + \varepsilon e^{i\theta}}\, d\theta$$ — 4M (1M for each correct parametrization)

$$= 2\pi i \times \text{Res}(f, e^{i\pi})$$

$$= 2\pi.$$

The first and third integrals combine to give

$$(1-(e^{2\pi i})^{-1/2})\int_\varepsilon^R \frac{dx}{(1+x)\sqrt{x}} = 2\int_\varepsilon^R \frac{dx}{(1+x)\sqrt{x}}.$$

1M for the real integral

For the second integral

$$\left|\int_{\Gamma_2} f\right| \le \frac{2\pi R^{1/2}}{R-1}, \quad \text{since } R>1,$$

1M for inequality

so that, $\lim_{R\to\infty}\int_{\Gamma_2} f = 0.$

1M for limit

For the fourth integral,

$$\left|\int_{\Gamma_4} f\right| \le \frac{2\pi\varepsilon^{1/2}}{1-\varepsilon}, \quad \text{since } \varepsilon<1,$$

1M for inequality

so that $\lim_{\varepsilon\to 0}\int_{\Gamma_4} f = 0.$

1M for limit

Hence, upon letting R get large and ε get small, we have

$$\int_0^\infty \frac{dx}{(1+x)\sqrt{x}} = \pi.$$

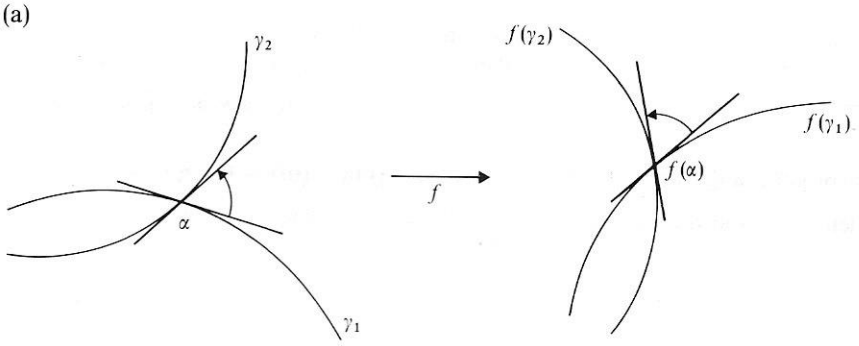
1M for conclusion

Total 17

PART III

Question 17

(a)



We may assume that $\gamma_1(0) = \gamma_2(0) = \alpha$.

Since $f'(\alpha) \neq 0$ and f' is continuous, it follows that $f' \neq 0$ on some neighbourhood of α. Now

$$(f\circ\gamma_1)'(t) = f'(\gamma_1(t))\gamma_1'(t),$$

so that for sufficiently small t we have

$$(f\circ\gamma_1)'(t) \neq 0.$$

Hence $f\circ\gamma_1$ is a smooth arc for t in some interval about 0. A similar argument applies to $f\circ\gamma_2$.

1M

The tangent to $f\circ\gamma_1$ at $f(\alpha)$ is

$$t \longrightarrow f(\alpha) + tf'(\alpha)\gamma_1'(0);$$

similarly for $f\circ\gamma_2$. The angle between γ_1 and γ_2 at α is

$$\operatorname{Arg}\gamma_2'(0) - \operatorname{Arg}\gamma_1'(0) \pmod{2\pi},$$

and lies in $[0, 2\pi)$.

1M

Similarly the angle between $f\circ\gamma_1$ and $f\circ\gamma_2$ at $f(\alpha)$ is

$$\operatorname{Arg}[f'(\alpha)\gamma_2'(0)] - \operatorname{Arg}[f'(\alpha)\gamma_1'(0)] \pmod{2\pi},$$

and lies in $[0, 2\pi)$.

1M

But

$$\begin{aligned}&\operatorname{Arg}[f'(\alpha)\gamma_2'(0)] - \operatorname{Arg}[f'(\alpha)\gamma_1'(0)]\\ &= \operatorname{Arg}f'(\alpha) + \operatorname{Arg}\gamma_2'(0) - \operatorname{Arg}f'(\alpha) - \operatorname{Arg}\gamma_1'(0) \pmod{2\pi}\\ &= \operatorname{Arg}\gamma_2'(0) - \operatorname{Arg}\gamma_1'(0) \pmod{2\pi}.\end{aligned}$$

1M

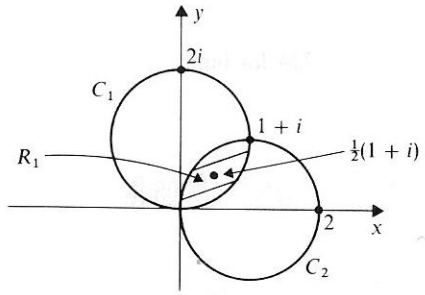
Thus the angle between $f \circ \gamma_1$ and $f \circ \gamma_2$ at $f(\alpha)$ differs from the angle between γ_1 and γ_2 at α by a multiple of 2π. Since both angles lie in $[0, 2\pi)$, they must be equal. 1M

(b) A *bilinear transformation* is one of the form

$$z \longrightarrow \frac{az + b}{cz + d} \qquad \text{where } ad - bc \neq 0.$$

A *conformal transformation* is one that preserves the angles between arcs. 1½M

(c) (i)



½M

At 0, the angle of intersection of C_1 and C_2 is $\frac{\pi}{2}$. ½M

(ii) Since $B(0) = 0$ and $B(1 + i) = \infty$, and since bilinear transformations map circles/lines into circles/lines, it follows that B maps C_1, C_2 onto two lines through 0.

Since bilinear transformations are conformal, $B(C_1)$ and $B(C_2)$ must meet at right angles at $0 = B(0)$; hence $B(R_1)$ must be a quarter-plane with vertex at the origin. 1M

Now

$$B(2) = \frac{2}{2 - (1 + i)} = \frac{2}{1 - i} = 1 + i, \quad \text{and}$$

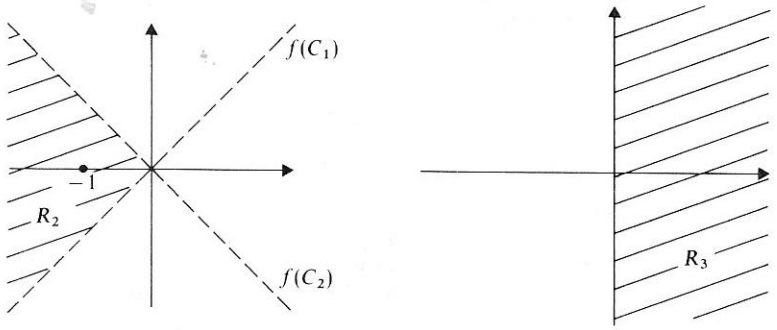
$$B(2i) = \frac{2i}{2i - (1 + i)} = \frac{2i}{i - 1} = 1 - i;$$

hence $f(C_1), f(C_2)$ are two lines through the origin at angles $\pm\frac{\pi}{4}$ with the direction of the positive real axis. These determine four quarter-planes, one of which is $B(R_1)$. 1M

Now $\frac{1}{2}(1 + i)$ belongs to R_1, and

$$B(\tfrac{1}{2}(1 + i)) = \frac{\frac{1}{2}(1 + i)}{\frac{1}{2}(1 + i) - (1 + i)} = -1;$$

and so, since this must belong to $B(R_1)$, $B(R_1)$ must be R_2. 1M



(iii) The function $z \longrightarrow z^2$ maps R_2 to R_3. 1M

(iv) This can be done by the bilinear transformation

$$b(z) = \frac{z - 1}{z + 1}.$$

For

$$\begin{cases} b(0) = -1, \\ b(i) = \dfrac{i - 1}{i + 1} = i, \quad \text{and} \\ b(-i) = -i. \end{cases}$$

Thus b maps the imaginary axis to $\{|z| = 1\}$, since it maps circles/lines to circles/lines. Also $b(1) = 0$, so that b must map R_3 onto $\{|z| < 1\}$. 3M

(v) First, $z \longrightarrow \dfrac{z}{z-(1+i)}$ maps $R_1 \longrightarrow R_2$;

next, $z \longrightarrow \left(\dfrac{z}{z-(1+i)}\right)^2$ maps $R_1 \longrightarrow R_3$;

and, finally, the mapping $R_1 \longrightarrow R_4$ is given by

$$z \longrightarrow \frac{\left(\dfrac{z}{z-(1+i)}\right)^2 - 1}{\left(\dfrac{z}{z-(1+i)}\right)^2 + 1}. \qquad 1\tfrac{1}{2}\text{M}$$

Comment: This last fraction can be expressed as

$$\frac{2(1+i)z - 2i}{2z^2 - 2(1+i)z + 2i},$$

though this is not necessary. *Total 16*

Question 18

(a) The function F defined by

$$F(p) = \int_0^\infty e^{-pt} f(t)\,dt \qquad 1\text{M}$$

is called the Laplace transform of f. The corresponding inversion formula is

$$f(t) = \frac{1}{2\pi i} \lim_{r\to\infty} \int_{\gamma-ir}^{\gamma+ir} e^{pt} F(p)\,dp \qquad 1\text{M for formula (This can be expressed in the form } \int_{\gamma-i\infty}^{\gamma+i\infty}\text{)}$$

where

(i) $F(p)$ is analytic on a region containing the half-plane $\operatorname{Re} p \geq x_0$; 1M

(ii) there exist real numbers $k > 1$ and K such that $|F(p)| < K|p^{-k}|$, for p in $\{p: \operatorname{Re} p \geq x_0\}$ and $|p|$ greater than some $|p_0|$; 1M for inequality, 1M for $|p|$ statement

(iii) $F(p)$ is real when p is real. 1M

(b) $\mathscr{L}[f'](p) = pF(p) - f(0) = pF(p)$, $\operatorname{Re} p > x_0$ for any $x_0 > 0$, 1M

$\mathscr{L}[f''](p) = p^2F(p) - pf(0) - f'(0)$, $\operatorname{Re} p > x_0$ for any $x_0 > 0$ 1M

$= p^2F(p) - 1.$

Hence

$$p^2F(p) - 1 + 2pF(p) - 3F(p) = \frac{1}{p^2},$$

and so

$$F(p) = \frac{p^2+1}{p^2(p+3)(p-1)}. \qquad 1\text{M}$$

(c) (i) $F(p)$ is analytic everywhere except at $p = 0$, $p = -3$, $p = 1$. Therefore it is analytic on the required region when $x_0 > 1$. 1M

(Note that the eventual solution function $f(t)$ is such that the formulae for $\mathscr{L}[f']$, $\mathscr{L}[f'']$ are valid and (i) is satisfied under the assumption that $\operatorname{Re} p > x_0$ for any $x_0 > 1$.)

(ii) $|F(p)| = \left|\dfrac{p^2+1}{p^2(p+3)(p-1)}\right|$

$$\leq \frac{|p^2+1|}{|p|^2||p|-3||p|-1|}.$$

Now for $|p| > 6$,

$$\frac{1}{|p|-3} < \frac{2}{|p|}$$

$$\frac{1}{|p|-1} < \frac{2}{|p|}$$

$$|p|^2 + 1 < 2|p|^2,$$

and so

$$|F(p)| \leq \frac{2|p|^2 4}{|p|^2|p|^2}$$

$$= \frac{8}{|p|^2}.$$

1M for correct inequality

So let $|p_0| = 6$, $k = 2$, $K = 8$, and condition (ii) holds.

1M for $|p_0|$, k and K

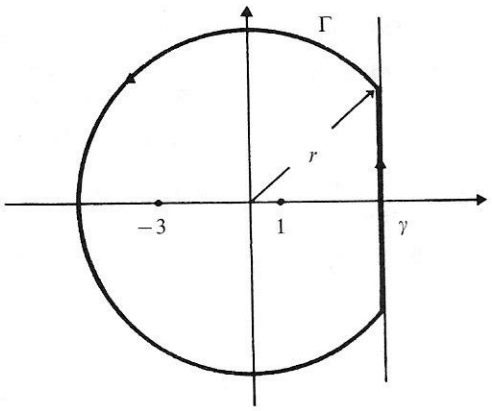
(iii) $F(p)$ is real when p is real.

1M

Hence the conditions for the inversion formula hold and

$$f(t) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} \frac{(p^2+1)e^{pt}}{p^2(p+3)(p-1)}\,dp.$$

Consider the closed contour Γ made up of the line Re $p = \gamma(>1)$ and an arc of a circle with centre the origin and radius $r > 3$.



The residue of $p \longrightarrow \dfrac{(p^2+1)e^{pt}}{p^2(p+3)(p-1)}$ at 1 is $\dfrac{e^t}{2}$, and at -3 is $\dfrac{-5}{18}e^{-3t}$. The residue at 0 is

1M for residues at 1 and -3

$$\lim_{p\to 0}\left[\frac{d}{dp}\frac{(p^2+1)e^{pt}}{(p+3)(p-1)}\right] = \frac{-3t-2}{9}.$$

1M for residue at 0

Hence

$$f(t) = \frac{e^t}{2} - \frac{5}{18}e^{-3t} - \frac{(3t+2)}{9}.$$

1M for answer

Total 16

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